3 Sobolev Spaces

Exercise 3.1. We will prove it by induction. The case $k = 1$ is trivial, indeed $f(x_1, x_2) = f_1(x_2)f_2(x_1)$ and

$$\|f\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{\mathbb{R}} |f_1(x_2)||f_2(x_1)|\, dx_1 dx_2 = \|f_1\|_{L^1(\mathbb{R})}\|f_2\|_{L^1(\mathbb{R})}.$$

Let now $k \geq 2$ and assume the estimate is true for k. We have to prove that the estimate still holds for $k+1$. Let us treat the case $k = 2$ to see how the proof of the induction hypothesis should go. We have

$$f(x) = f_1(x_2, x_3)f_2(x_1, x_3)f_3(x_1, x_2).$$

Thanks to Fubini's theorem and Cauchy-Schwarz inequality (used each twice), we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3} |f(x)|dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f_1(x_2, x_3)||f_2(x_1, x_3)||f_3(x_1, x_2)|dx_1\, dx_2\right) dx_3 \\
&\leq \|f_3\|_{\mathrm{L}^2(\mathbb{R}^2)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f_1(x_2, x_3)|^2 |f_2(x_1, x_3)|^2 dx_1\, dx_2\right)^{\frac{1}{2}} dx_3 \\
&= \|f_3\|_{\mathrm{L}^2(\mathbb{R}^2)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_1(x_2, x_3)|^2 dx_2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f_2(x_1, x_3)|^2 dx_1\right)^{\frac{1}{2}} dx_3 \\
&\leq \|f_3\|_{\mathrm{L}^2(\mathbb{R}^2)} \left(\int_{\mathbb{R}} \|f_1(\,\cdot, x_3)\|^2_{\mathrm{L}^2(\mathbb{R})}\, dx_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|f_2(\,\cdot\,, x_3)\|^2_{\mathrm{L}^2(\mathbb{R})}\, dx_3\right)^{\frac{1}{2}} \\
&= \|f_1\|_{\mathrm{L}^2(\mathbb{R}^2)} \|f_2\|_{\mathrm{L}^2(\mathbb{R}^3)} \|f_3\|_{\mathrm{L}^2(\mathbb{R}^2)}.
\end{aligned}$$

For every fixed $x_{k+2} \in \mathbb{R}$, by Hölder inequality with $p = k+1, p' = \frac{k+1}{k}$ we get

$$\int_{\mathbb{R}^{k+1}} |f(x)|dx_1 \dots dx_{k+1} \leq \|f_{k+2}\|_{L^{k+1}(\mathbb{R}^{k+1})} \left(\int_{\mathbb{R}^{k+1}} \prod_{i=1}^{k+1} |f_i|^{\frac{k+1}{k}} dx_1 \dots dx_{k+1}\right)^{\frac{k}{k+1}}. \tag{1}$$

By applying the inductive hypothesis on the functions $|f_i|^{\frac{k+1}{k}}$ we obtain

$$\int_{\mathbb{R}^{k+1}} \prod_{i=1}^{k+1} |f_i|^{\frac{k+1}{k}} dx_1 \dots dx_{k+1} \leq \prod_{i=1}^{k+1} \|f_i\|^{\frac{k+1}{k}}_{L^{k+1}(\mathbb{R}^k)},$$

which together with (1), gives

$$\int_{\mathbb{R}^{k+1}} |f(x)|dx_1 \dots dx_{k+1} \leq \|f_{k+2}\|_{L^{k+1}(\mathbb{R}^{k+1})} \prod_{i=1}^{k+1} \|f_i\|_{L^{k+1}(\mathbb{R}^k)}. \tag{2}$$

Since by hypothesis $f_i \in L^{k+1}(\mathbb{R}^{k+1})$, all the functions $x_{k+2} \mapsto \|f_i\|_{L^{k+1}(\mathbb{R}^k)}$ belong to $L^{k+1}(\mathbb{R})$. Thus by applying again Hölder inequality (with $k+1$ terms) we get that $\prod_{i=1}^{k+1} \|f_i\|_{L^{k+1}(\mathbb{R}^k)} \in L^1(\mathbb{R})$ and so, by integrating (2) in dx_{k+2} we conclude

$$\int_{\mathbb{R}^{k+2}} |f(x)| dx_1 \dots dx_{k+2} \leq \prod_{i=1}^{k+2} \|f_i\|_{L^{k+1}(\mathbb{R}^{k+1})}.$$

Exercise 3.2. — By applying Hölder inequality with exponents $\frac{p}{\alpha r}$ and $\frac{q}{(1-\alpha)r}$ we get

$$\int_\Omega |f|^r = \int_\Omega |f|^{\alpha r} |f|^{(1-\alpha)r} \leq \left(\int_\Omega |f|^p\right)^{\frac{\alpha r}{p}} \left(\int_\Omega |f|^q\right)^{\frac{(1-\alpha)r}{q}},$$

from which we deduce the desired inequality.

— For every $x, y \in \Omega$ we have

$$\frac{|u(x)-u(y)|}{|x-y|^\gamma} = \frac{|u(x)-u(y)|^t}{|x-y|^{t\alpha}} \frac{|u(x)-u(y)|^{1-t}}{|x-y|^{(1-t)\beta}} \leq [u]^t_{C^{0,\alpha}(\Omega)} [u]^{1-t}_{C^{0,\beta}(\Omega)},$$

and by taking the supremum all over $x, y \in \Omega$ we get the thesis.

Exercise 3.3. Since $u(x) \to +\infty$ as $|x| \to 0$, we deduce $u \notin L^\infty(\Omega)$. Moreover, for every $q < \infty$ we have

$$\int_\Omega |u(x)|^q \, dx = C \int_0^{\frac{1}{e}} |\log(-\log r)|^q \, r^{d-1} \, dr.$$

By the change of variables $-\log r = s$ we get that this last integral is equal to

$$\int_1^\infty (\log s)^q \, e^{-ds} \, ds < \infty.$$

Moreover, its distributional gradient is given by

$$\nabla u(x) = \frac{x}{|x|^2 \log |x|}, \tag{3}$$

and we get, by the same change of coordinates as before,

$$\int_\Omega |\nabla u(x)|^d \, dx = \int_\Omega \frac{1}{|x|^d |\log|x||^d} \, dx = \beta(d) \int_0^{\frac{1}{e}} \frac{1}{r|\log r|^d} \, dr = \beta(d) \int_1^\infty \frac{1}{s^d} \, ds < \infty.$$

Thus $u \in W^{1,d}(\Omega) \setminus L^\infty(\Omega)$.

Remark. clearly (3) holds in the sense of distributions on the domain $\Omega \setminus \{0\}$. In order to prove that the singularity at zero does not cause any problem, we proceed as follows : let $\chi \in C^\infty([0,+\infty),[0,1])$ be decreasing with $\chi(s) = 1$ for every $s \in [0,1]$ and $\chi(s) = 0$ for every $s \geq 2$. For every $\varepsilon > 0$ let $\chi_\varepsilon(s) = \chi(s/\varepsilon)$. Then for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned} \int_\Omega u\nabla\varphi dx =& \lim_{\varepsilon\to 0}\int_\Omega (1-\chi_\varepsilon(|x|))u\nabla\varphi dx \\ =& -\lim_{\varepsilon\to 0}\int_\Omega (1-\chi_\varepsilon(|x|))\varphi\nabla u dx + \lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon'(|x|)\frac{x}{|x|}u\varphi dx. \end{aligned} \tag{4}$$

The first limit in the right hand side is equal to $\int_\Omega \varphi\nabla u\, dx$ by dominated convergence theorem, since ∇u given in (3) belongs to $L^1(\Omega)$. The second limit is equal to 0, since

$$\left|\chi_\varepsilon'(|x|)\frac{x}{|x|}u\varphi\right| \leq \log\left(\log\left(\frac{1}{\varepsilon}\right)\right)\frac{1}{\varepsilon}\|\varphi\|_{L^\infty}\mathbf{1}_{[\varepsilon,2\varepsilon]}(|x|)$$

and therefore its integral (for $n \geq 2$) converges to 0 as $\varepsilon \to 0$.

Exercise 3.4. Just take in dimension $d = 1$ the open interval $\Omega = (1,\infty)$ and $u(x) = x^{-\frac{1}{q}}$. Since $p > q$, we have $u \in W^{1,p}(\Omega)$, but

$$\int_\Omega |u(x)|^q\, dx = \int_1^\infty \frac{1}{x}\, dx = \infty.$$

Exercise 3.5. If $p > n$, then $W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$. Thus $fg \in L^p(\mathbb{R}^d)$ since

$$\|fg\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \|f\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\|g\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

Moreover, since $\partial_i(fg) = \partial_i fg + f\partial_i g$, by an analogous reasoning, we get $\partial_i(fg) \in L^p(\mathbb{R}^d)$, which shows that $fg \in W^{1,p}(\mathbb{R}^d)$.